

Primal and Dual Bounds for Optimal Transmission Switching

Carleton Coffrin[†], Hassan L. Hijazi*, Karsten Lehmann*, and Pascal Van Hentenryck*

*NICTA & The Australian National University, Canberra, ACT 2601, Australia

Email: {hassan.hijazi,karten.lehmann,pvh}@nicta.com.au

[†]NICTA, Melbourne, VIC 3010, Australia

Email: carleton.coffrin@nicta.com.au

Abstract—It has been suggested that Optimal Transmission Switching (OTS) may reduce generator dispatch costs by as much as 10%, saving millions of dollars annually. However, this conclusion has been deduced primarily from studies using the DC power flow approximation on two power networks derived from the IEEE 118-bus and RTS-96 cases. This paper is a systematic study of the OTS problem. Various OTS formulations are considered for computing primal and dual bounds on a variety of power networks. The results demonstrate that the DC power flow model is inadequate for solving the OTS problem, and that mixed-integer nonlinear optimization techniques are instrumental in finding high-quality primal and dual bounds. The paper also indicates that, on a variety of benchmarks, transmission line switching may bring generation cost reductions between 0% to 29%.

Keywords—Optimal Transmission Switching, Optimal Power Flow, AC Power Flow, DC Power Flow, Mixed-Integer Nonlinear Programming

NOMENCLATURE

\tilde{I}	AC current
$\tilde{V} = v + i\theta$	AC voltage
$\tilde{S} = p + iq$	AC power
$\tilde{Z} = r + ix$	Line resistance
$\tilde{Y} = g + ib$	Line admittance
$\tilde{V} = \tilde{V} \angle\theta^\circ$	Polar form
\mathcal{PN}	Power network
N	Set of buses in a power network
$N(n)$	Set of buses connected to bus n by a line
L	Set of lines $\langle n, m \rangle$ in a power network where n is the from node
L^r	Set of lines $\langle n, m \rangle$ in a power network where n is the to node
s	Slack bus
$\bar{\theta}_\Delta$	Maximum phase angle difference
μ	Average
\bar{x}	Upper bound of x
\underline{x}	Lower bound of x
$\Re(\cdot)$	Real part of a complex number
$\Im(\cdot)$	Imaginary part of a complex number
$(\cdot)^*$	conjugate of a complex number

I. INTRODUCTION

It is well-known that changing the topology of a power network can reduce, or eliminate, congestion created by line thermal limits or nominal voltage requirements [1], [2]. More

recently, it has been observed that topology control may lead to cost savings around 10% in locational marginal price energy markets [3], [4], [5], [6], [7]. Topology control for reducing generation costs was originally suggested in [8] and formalized in [3], and is often called Optimal Transmissions Switching (OTS).

From a mathematical standpoint, the OTS problem presents a challenging non-convex Mixed-Integer NonLinear Program (MINLP). In response to these computational challenges, many studies [3], [4], [5], [7], [9], [10], [11], [12] approximate the OTS problem with the widely used DC power flow model. This reduces the OTS problem to a Mixed-Integer Linear Program (MIP). However, despite its widespread adoption, the accuracy and applicability of the DC power flow model is an active point of discussion: Some papers take an optimistic outlook (e.g., [13], [14]), while others are more reserved (e.g., [15], [16]). In the context of the OTS problem, it was said, “there is no guarantee that an AC feasible solution will be obtained when using an AC approximate formulation to produce an initial solution and, thus future research is needed to investigate this concern.” [17] Heading this call, recent works [6], [12], [18] approach the OTS problem with AC power flow models instead of the DC model.

The goals of this paper are twofold: (1) to evaluate and bound the potential gains of OTS in the AC context on a variety of networks; (2) to find effective solution methods for solving the OTS problem in the AC context. These goals are achieved through three core technologies:

- 1) A novel highly tractable flow relaxation (NF-OTS) to bound the benefits of OTS;
- 2) A stronger, computationally intensive, convex quadratic relaxation proposed in [19] to determine the feasibility of network topologies and provide tight OTS bounds;
- 3) A heuristic branch and bound algorithm implemented in the solver BONMIN for convex MINLPs [20] to find high-quality solutions to the OTS problem.

The key findings of the paper can be summarized as follows:

- 1) Line switching may bring benefits on congested networks, but not on standard MATPOWER benchmarks.
- 2) The DC model does not appear to be appropriate for OTS studies as it exhibits significant AC-feasibility issues while both underestimating and overestimating the benefits of line switching in different contexts.

- 3) The NF-OTS relaxation, despite its simplicity, provides tight dual bounds on standard MATPOWER benchmarks. However, its quality deteriorates on congested benchmarks. The QC relaxation produces tight lower bounds on all benchmarks, albeit at a higher computational cost.
- 4) The heuristic branch and bound algorithm implemented in BONMIN consistently produces high-quality solutions to the OTS problem.

The rest of the paper is organized as follows. Section II introduces the OTS problem using the AC power flow equations and discusses various solution methods. Section III presents a simple case study to illustrate the strengths and weaknesses of the solution methods. Section IV studies the potential benefits of line switching using the NF-OTS relaxation on a wide variety of networks. Section V analyzes the behavior of the DC model for transmission switching. Section VI reports the use of BONMIN-OTS to find primal solutions to the OTS problem, demonstrating the benefits of line switching on a number of test cases, and Section VII concludes the paper.

II. OPTIMAL TRANSMISSION SWITCHING

Optimal Transmission Switching (OTS) is a natural extension of the classical Optimal Power Flow (OPF) problem, that includes topological changes. An OTS formulation based on the AC power flow equations is presented in Model 1. The input data and variables are described in the model and the set of constraints is discussed here. The objective (M1.1) minimizes the total cost of generation modeled as a quadratic function. Constraint (M1.2) fixes the slack-bus's imaginary component to 0, breaking rotational symmetries and allow solutions comparison across different models. Constraints (M1.3) enforce the voltage magnitude limits. Constraints (M1.4–M1.5) ensure conservation of energy (i.e., Kirchhoff's Current Law) and constraints (M1.6–M1.7) capture the flow of power via Ohm's Law. The discrete indicator variable $z_{nm} \in \{0, 1\}$ allows line (n, m) to be added or removed from the network: if $z_{nm} = 1$, the power flows by Ohm's Law; If $z_{nm} = 0$, no power flows across the line. Constraints (M1.8) capture the thermal limit of the lines and constraints (M1.9–M1.10) are redundant constraints that capture line losses. The AC-OTS problem is a challenging non-convex Mixed-Integer Non-Linear Program (MINLP) outside the scope of current global optimization solvers. Both Couenne [21] and SCIP [22] have difficulty solving the continuous AC-OPF problem, and hence the OTS extension is out of reach.

Due to the computational challenges of solving the AC-OTS globally, one must resort to solving an alternate version of the problem that is computationally tractable. There are three main approaches in making the AC-OTS more tractable: (1) heuristics; (2) approximations of the power flow equations; and (3) relaxations of the power flow equations. Heuristics are often very fast to compute, but provide no quality guarantees. Approximations can reduce the computational complexity, but can return infeasible solutions in the original space [17], [12]. Relaxations provide provable dual bounds to the solution of the original problem. In the context of AC-OTS, such relaxations make it possible to bound the potential benefits of line switching. Equally important, relaxations may be used to prove that there is no solution for a particular network

Model 1 The AC-OTS Problem

Inputs:

$\langle N, L, s \rangle$	- the power network
c_n^u, c_n^l, c_n	- cost coefficients for generator n
$ \tilde{V}_n , \tilde{V}_n^* $	- voltage limits for bus n
$\underline{p}_n^g, \bar{p}_n^g$	- active injection limits for bus n
$\underline{q}_n^g, \bar{q}_n^g$	- reactive injection limits for bus n
$\underline{p}_n^l, \bar{p}_n^l$	- demands at bus n
$\underline{q}_n^l, \bar{q}_n^l$	- demands at bus n
g_{nm}, b_{nm}	- admittance of line nm
$ \tilde{S}_{nm} $	- thermal limit of line nm

Variables:

\tilde{V}_n	- complex voltage on bus n
$p_n^g \in (\underline{p}_n^g, \bar{p}_n^g)$	- active generation on bus n
$q_n^g \in (\underline{q}_n^g, \bar{q}_n^g)$	- reactive generation on bus n
$p_{nm} \in (- \tilde{S}_{nm} , \tilde{S}_{nm})$	- active flow on line nm
$q_{nm} \in (- \tilde{S}_{nm} , \tilde{S}_{nm})$	- reactive flow on line nm
$p_{nm}^+ \in (0, \infty)$	- active losses on line nm
$q_{nm}^+ \in (-\infty, \infty)$	- reactive losses on line nm
$z_{nm} \in \{0, 1\}$	- indicator variable for line nm

Minimize:

$$\sum_{i \in N} c_i^u (p_i^g)^2 + c_i^l (p_i^g) + c_i \quad (\text{M1.1})$$

Subject To:

$$\Im(\tilde{V}_s) = 0 \quad (\text{M1.2})$$

$$|\tilde{V}_n|^2 \leq \Re(\tilde{V}_n \tilde{V}_n^*) \leq |\tilde{V}_n^*|^2 \quad \forall n \in N \quad (\text{M1.3})$$

$$p_n^l - p_n^g = \sum_{m \in N(n)} p_{nm} \quad \forall n \in N \quad (\text{M1.4})$$

$$q_n^l - q_n^g = \sum_{m \in N(n)} q_{nm} \quad \forall n \in N \quad (\text{M1.5})$$

$$\forall \langle n, m \rangle \in L \cup L^r$$

$$p_{nm} = z_{nm} (g_{nm} \tilde{V}_n \tilde{V}_n^* - g_{nm} \Re(\tilde{V}_n \tilde{V}_m^*) - b_{nm} \Im(\tilde{V}_n \tilde{V}_m^*)) \quad (\text{M1.6})$$

$$q_{nm} = z_{nm} (-b_{nm} \tilde{V}_n \tilde{V}_n^* + b_{nm} \Re(\tilde{V}_n \tilde{V}_m^*) - g_{nm} \Im(\tilde{V}_n \tilde{V}_m^*)) \quad (\text{M1.7})$$

$$p_{nm}^2 + q_{nm}^2 \leq |\tilde{S}_{nm}|^2 \quad (\text{M1.8})$$

$$p_{nm} + p_{mn} = p_{nm}^+ \quad (\text{M1.9})$$

$$q_{nm} + q_{mn} = q_{nm}^+ \quad (\text{M1.10})$$

topology. Indeed, a relaxation's solutions are a superset of the original feasible set. Dual bounds and infeasibility proofs are outside the scope of heuristics and approximations. The rest of this section introduces various heuristics, relaxations, and approximations of the AC-OTS problem.

A. DC Power Flow Approximation

The DC power flow model is a popular approximation of the AC power flow model. It starts by considering the Constraints (M1.6–M1.7) in polar form,

$$\Re(\tilde{V}_n \tilde{V}_n^*) = |\tilde{V}_n|^2 \quad (1)$$

$$\Re(\tilde{V}_n \tilde{V}_m^*) = |\tilde{V}_n| |\tilde{V}_m| \cos(\theta_n^\circ - \theta_m^\circ) \quad (2)$$

$$\Im(\tilde{V}_n \tilde{V}_m^*) = |\tilde{V}_n| |\tilde{V}_m| \sin(\theta_n^\circ - \theta_m^\circ) \quad (3)$$

It then makes the following assumptions: (1) the susceptance is large relative to the conductance $|g| \ll |b|$; (2) the phase angle difference is small enough to ensure $\sin(\theta_n^\circ - \theta_m^\circ) \approx \theta_n^\circ - \theta_m^\circ$ and $\cos(\theta_n^\circ - \theta_m^\circ) \approx 1.0$; and (3) the voltage magnitudes $|\tilde{V}|$

are close to 1.0 and do not vary significantly. Under these assumptions, equations (M1.6–M1.7) reduce to

$$p_{nm} = z_{nm}(-b_{nm}(\theta_n^\circ - \theta_m^\circ)) \quad (4)$$

To obtain a DC formulation of the OTS problem (DC-OTS), equation (4) can be written using a classic *big-M* formulation to produce a Mixed-Integer Linear Program (MIP), which is computationally appealing. In fact, the bulk of work on the OTS problem has focused on this DC-OTS variant [3], [4], [5], [7], [9], [10], [11]. Its primary drawback is that it approximates the power flow equations and is not a relaxation. This means that results on the DC-OTS problem cannot be applied directly to the AC-OTS problem, as mentioned in [17].

B. Network Flow Relaxation

The network flow relaxation utilizes a tight relaxation of line losses providing a lower bound to the OTS problem that can be computed very efficiently. Since

$$p_{nm}^+ = r|\tilde{I}|^2 = r\frac{|\tilde{S}_{nm}|^2}{|\tilde{V}_n|^2} = r\frac{p_{nm}^2 + q_{nm}^2}{|\tilde{V}_n|^2} \quad (5)$$

$$q_{nm}^+ = x|\tilde{I}|^2 = x\frac{|\tilde{S}_{nm}|^2}{|\tilde{V}_n|^2} = x\frac{p_{nm}^2 + q_{nm}^2}{|\tilde{V}_n|^2}, \quad (6)$$

a convex relaxation of (5)-(6) can be obtained by transforming them into inequalities:

$$p_{nm}^+|\tilde{V}_n|^2 \geq r_{nm}(p_{nm}^2 + q_{nm}^2) \quad (7)$$

$$q_{nm}^+|\tilde{V}_n|^2 \geq x_{nm}(p_{nm}^2 + q_{nm}^2) \quad (8)$$

The complete relaxation is identical to Model M1 where Ohm's Law constraints (M1.6–M1.7) are ignored and line loss constraints (7)-(8) are enforced. This model is denoted by NF-OTS.

Observe that NF-OTS is a relaxation of both the AC-OPF and AC-OTS. Indeed, for any disconnected line $\langle n, m \rangle$, the values p_{nm}, p_{mn}, q_{nm} and q_{mn} can be set to zero, satisfying the loss constraints (7) and (8) without influencing other variables in the Kirchhoff's Current Law constraints (M1.4–M1.5).

C. The QC Relaxation

The QC relaxation was introduced in [19] to provide high-quality dual bounds to a variety of power flow problems. This relaxation convexifies the trigonometric functions and variable products involved in the polar form using quadratic and linear relaxations. Let $\tilde{v}_n = |\tilde{V}_n|$, the QC relaxation is defined as:

$$p_{nm} = g_{nm}\tilde{v}_n - g_{nm}\tilde{w}c_{nm} - b_{nm}\tilde{w}s_{nm} \quad (9a)$$

$$q_{nm} = -b_{nm}\tilde{v}_n + b_{nm}\tilde{w}c_{nm} - g_{nm}\tilde{w}s_{nm} \quad (9b)$$

$$\tilde{v}_n \in \langle \tilde{v}_n^2 \rangle^R \quad (9c)$$

$$\check{c}_{nm} \in \langle \cos(\theta_n^\circ - \theta_m^\circ) \rangle^R \quad (9d)$$

$$\check{s}_{nm} \in \langle \sin(\theta_n^\circ - \theta_m^\circ) \rangle^R \quad (9e)$$

$$\check{w}_{nm} \in \langle \tilde{v}_n, \tilde{v}_m \rangle^M \quad (9f)$$

$$\check{w}c_{nm} \in \langle \check{w}_{nm}, \check{c}_{nm} \rangle^M \quad (9g)$$

$$\check{w}s_{nm} \in \langle \check{w}_{nm}, \check{s}_{nm} \rangle^M \quad (9h)$$

$$(7) - (8)$$

Where

$$\langle \tilde{v}^2 \rangle^R = \begin{cases} \tilde{v}\tilde{v} \geq \tilde{v}^2 \\ \tilde{v}\tilde{v} \leq (\bar{v} + \underline{v})\tilde{v} - \bar{v}\underline{v} \end{cases}$$

$$\langle \cos(\theta_\Delta) \rangle^R = \begin{cases} \check{c} \leq 1 - \frac{1 - \cos(\bar{\theta}_\Delta)}{(\bar{\theta}_\Delta)^2} \theta_\Delta^2 \\ \check{c} \geq \cos(\bar{\theta}_\Delta) \end{cases}$$

$$\langle \sin(\theta_\Delta) \rangle^R = \begin{cases} \check{s} \leq \cos(\bar{\theta}_\Delta/2)(\theta_\Delta - \bar{\theta}_\Delta/2) + \sin(\bar{\theta}_\Delta/2) \\ \check{s} \geq \cos(\bar{\theta}_\Delta/2)(\theta_\Delta + \bar{\theta}_\Delta/2) - \sin(\bar{\theta}_\Delta/2) \end{cases}$$

$$\langle vw \rangle^M = \begin{cases} \check{v}\check{w} \geq vw + \underline{v}v - \underline{v}\underline{w} \\ \check{v}\check{w} \geq \bar{v}v + \bar{w}v - \bar{v}\bar{w} \\ \check{v}\check{w} \leq \underline{v}w + \underline{w}v - \underline{v}\underline{w} \\ \check{v}\check{w} \leq \bar{v}w + \bar{w}v - \bar{v}\bar{w} \end{cases}$$

QC-based relaxations of the AC-OPF and AC-OTS are described in [19] and denoted by QC-OPF and QC-OTS in this paper.

D. The SDP and SOCP Relaxations

A Positive Semi-Definite Programming (SDP) relaxation of the AC-OPF was proposed in [23]. This relaxation uses a lift-and-project idea to convexify the problem by introducing the following W variables for the voltage product terms,

$$\Re(\tilde{V}_n \tilde{V}_n^*) = W_n \quad (10)$$

$$\Re(\tilde{V}_n \tilde{V}_m^*) = \Re(W_{nm}) \quad (11)$$

$$\Im(\tilde{V}_n \tilde{V}_m^*) = \Im(W_{nm}) \quad (12)$$

and adding a constraint that the matrix formed by all of the W variables should be positive semi-definite, $W \geq 0$. Noting that the positive semi-definite constraint does not exploit the sparsity of power networks, a Second Order Cone Program (SOCP) simplification has been suggested [24]. This is achieved by replacing $W \geq 0$ with the following constraint for each line $\langle n, m \rangle \in L$,

$$W_n W_m \geq \Re(W_{nm})^2 + \Im(W_{nm})^2 \quad (13)$$

Both of these formulations have been studied extensively in the context of AC-OPF, so it is useful to investigate their applicability to the AC-OTS problem. We refer to these models as the SDP-OPF and SOCP-OPF in this paper.

E. The Bonmin Heuristic

BONMIN [20] is a generic solver designed for MINLPs featuring convex constraints. When this convexity assumption holds, BONMIN produces globally optimal solutions. In the case of the AC-OTS problem, the non-linear constraints being non-convex, BONMIN is not guaranteed to find optimal solutions, however, it can be used as a heuristic (BONMIN-OTS) for finding high-quality solutions to the AC-OTS problem.

III. A 3-BUS CASE STUDY

The previous section introduced the AC-OTS problem as well as several relaxations and approximations of the problem. This section uses a simple 3-bus network as an illustration of some of the advantages and shortcomings of the various models. The three-bus power system is depicted in Figure 1 and the network parameters are presented in Table I. This network is designed with a distant cheap generator at bus 1 and

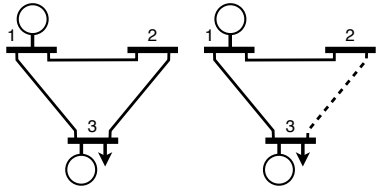


Fig. 1. 3-Bus Example Network Diagram, closed (left) and open (right) configurations.

TABLE I. THREE-BUS SYSTEM NETWORK DATA (100MVA BASE).

Bus Parameters					Line Parameters				
bus	p^l	q^l	$ \tilde{V} $	\tilde{V}	From-To bus	r	x	b	$ S $
1	0	0	0.9	1.1	1-2	0.00	0.05	0	∞
2	0	0	0.9	1.1	2-3	0.00	0.05	0	∞
3	100	0	0.9	1.1	1-3	0.10	0.10	0	∞

Generator Parameters						
Generator	p^g, \bar{p}^g	q^g, \bar{q}^g	c''	c'	c	
1	$0, \infty$	$-\infty, \infty$	0	1	0	
3	$0, \infty$	0, 0	0	10	0	

an expensive generator at bus 3 which is co-located with the load. The cheapest solution delivers as much power as possible from generator 1 and as little as possible from generator 3. Based on the network demand of 100 MW, we observe that the cost of any AC-OPF solution is within the range of \$100–\$1,000.

Figure 1 depicts a closed and open configuration of the network used to investigate the effects of topology changes. AC-OPF solutions to both of these configurations are presented in Tables II–III. Without any binding constraints, both configurations serve all of the load from the distant generator with a cost of \$101 and \$110 respectively, the small increase above \$100 is due to line losses in both cases. At this point, adding lines to the network decreases the aggregate network resistance leading to cheaper power transmissions, confirming a common intuition [6]. However, the solutions get more interesting in both topologies when binding network constraints (i.e. congestion) are introduced.

We consider three congestion scenarios for this 3-bus example: (1) line capacity congestion by reducing the thermal limit of line 2–3 to 1 MVA; (2) voltage bound congestion by reducing the voltage range from ± 0.1 to ± 0.02 ; and (3) both line capacity and voltage congestions simultaneously. Table IV reports the cheapest generator dispatches across the different scenarios, topologies, and power flow models. We can make the following observations: the DC-OPF captures line capacity congestion well, but does not capture voltage congestion effects; The various power flow relaxations (SDP, QC, SOCP, NF) form a gradient of different accuracies, which correlate with computational difficulty. Let us emphasize that, unlike other relaxations, the quality of the lower bound produced by QC-OTS may be improved when tightening variables bounds as shown in this table, where the model was optimized with two different bounds on the phase angle difference, of 5° and 15° .

Table V presents optimal OTS solutions produced with the various models for the congested scenarios. The results show that the NF-OTS and DC-OTS may significantly underestimate

TABLE II. THREE-BUS SYSTEM SOLUTION (CLOSED CONFIGURATION).

Bus Values					Line Values		
Bus	p^g	q^g	$ \tilde{V} $	θ° , deg.	Line	From MVA	To MVA
1	101	5	1.100	0.000	1-2	64	63
2	0	0	1.090	-1.434	2-3	63	63
3	0	0	1.080	-2.895	1-3	44	44

TABLE III. THREE-BUS SYSTEM SOLUTION (OPEN CONFIGURATION).

Bus Values					Line Values		
Bus	p^g	q^g	$ \tilde{V} $	θ° , deg.	Line	From MVA	To MVA
1	110	10	1.100	0.000	1-2	0	0
2	0	0	1.100	0.000	2-3	—	—
3	0	0	0.995	-5.243	1-3	110	100

TABLE IV. OPF RESULTS ON THE THREE-BUS SYSTEM UNDER DIFFERENT POWER MODELS.

	Capacity Congestion		Voltage Congestion		Both	
	closed	open	closed	open	closed	open
AC-OPF	985	110	102	655	985	655
SDP-OPF	972	110	102	655	972	655
QC-OPF (5°)	772	110	102	655	939	655
QC-OPF (15°)	529	110	101	655	748	655
SOCP-OPF	109	110	100	655	643	655
NF-OPF	109	110	100	112	111	112
DC-OPF	986	100	100	100	986	100

TABLE V. OTS RESULTS ON THE THREE-BUS SYSTEM UNDER DIFFERENT POWER MODELS.

Congestion Type	AC-OTS	SDP	QC	SOCP	NF	DC
Capacity	110	110	110	109	109	100
Voltage	102	102	101	100	100	100
Both	655	655	655	643	111	100

the cost of AC-OTS solutions. In fact, the gap between AC-OTS solutions and the bounds produced by NF-OTS and DC-OTS models can grow arbitrarily large, by increasing the relative costs of generators 1 and 3.

This example is of course contrived and such extreme cases are unlikely to occur in practice. Hence, in the rest of the paper, we consider standard test cases and study how the various models behave in these settings. Based on the previous case study, the rest of the paper uses AC-OPF, DC-OTS, and BONMIN-OTS for providing primal bounds to the AC-OTS problem, and NF-OTS and QC-OTS for providing fast-and-coarse or slow-and-accurate dual bounds respectively. All experiments were run on an AMD Opteron 4226, 2.7GHz with 64GB of memory using IPOPT 3.10.3 [25], BONMIN 1.6, and Cplex 12.6.0.

IV. THE POTENTIAL BENEFITS OF LINE SWITCHING

Before studying the OTS problem in detail, we first review the standard power network test cases and examine the potential benefits of line switching using the NF-OTS relaxation.

A. Standard Test Cases

The two most popular power system test case archives are the IEEE load flow cases [26] and the MATPOWER OPF cases [27]. The IEEE cases lack some of the necessary parameters for studying both the DC-OTS and the AC-OTS problems, including line thermal limit and generation cost data. Incorporating the additional required data, the research

community has developed modified versions of the IEEE 118-bus [28] and IEEE RTS-96 [11] networks, which we refer to as 118-OTS and RTS-96-OTS.

Some DC-OTS studies have adopted the original 118-OTS case [3], [5], [4], while others have made hybrids by combining parts of the IEEE 118-bus load flow case and the 118-OTS case [6], [7], [18]. In this paper, we strive to stay as close to the 118-OTS case as possible. Unfortunately, a QC-OPF study of 118-OTS indicated that there is no AC-OPF solution to this network. To make the 118-OTS case from [28] AC feasible, we perform the following modifications: (1) the transformer and synchronous condenser parameters from the MATPOWER 118-bus were integrated; (2) the generator at bus 89 was reintroduced as a synchronous condenser, using the reactive injection bounds of the MATPOWER case; (3) the active power demand of 440MW at bus 90 was reduced to 350MW. This 118-OTS-AC network is AC feasible and has 118 buses, 54 generators (of which 35 are synchronous condensers), and 187 lines (of which 9 are transformers). Its total active generation capacity is 5759 WM and the total demand in the system is 4519 WM. An AC-OPF solution to the network was obtained with IPOPT in 0.25 second and has a total generation cost of \$1965.

Other DC-OTS studies have used the RTS-96-OTS case [9], [10], [11], [12]. This network is considered at a variety of loading levels to simulate daily and seasonal changes in the demand profile [29], and typically only the active power demands are scaled while the reactive power demands remain constant. In its default configuration (i.e. 100% loading), this RTS-96-OTS network has 73 buses, 99 generators (of which 3 are synchronous condensers), and 117 lines (of which 15 are transformers). Its total active generation capacity is 10215 WM and the total demand in the system is 8550 WM. There is some variability in the generator cost functions used to study this network, here we adopt the same functions as [12]. AC-OPF feasibility is a challenge in the RTS-96-OTS case [12], and similar to previous work, we were unable to find an AC-OPF solution to this network in its default configuration. However, we did observe that by removing four lines from the network (110–111, 110–112, 210–211, 310–311), it can be made AC-OPF feasible. The symmetry of these lines is not surprising as the RTS-96-OTS network is simply three copies of the RTS-79 network linked together [29]. Feasible AC-OPF solutions to this network at the various loading levels were obtained with IPOPT in less than one second in all cases, and the resulting generation costs are presented in Table VI.

B. Line Switching on Standard Test Cases

This section explores the potential of line switching on the standard test cases. On the 118-OTS-AC case, the NF-OTS model provides a lower bound of \$1463 to the AC-OTS problem (a potential benefit of 25.5%). On the RTS-96-OTS case, Table VI reports the relative quality between the AC-OPF feasible solution and the NF-OTS model, showing potential benefits from 5.62% to 31.23%. All of these results indicate the significant potential of OTS, however it is important to note that these can be very optimistic estimates, as demonstrated in Section III.

Consider now the MATPOWER OPF cases. Table VII reports the relative quality between MATPOWER's AC-OPF

TABLE VI. OTS POTENTIAL ON THE RTS-96-OTS CASE.

Loading	AC-OPF	NF-OTS	Gap	Time (sec.)
100%	202222	186653	7.70%	< 1
95%	159069	150136	5.62%	< 1
90%	122205	113961	6.75%	< 1
85%	86698	78317	9.67%	< 1
80%	60976	45130	25.99%	< 1
75%	41103	34307	16.53%	< 1
70%	34582	23782	31.23%	< 1

TABLE VII. OTS POTENTIAL ON NINE MATPOWER BENCHMARKS.

Benchmark	AC-OPF	NF-OTS	Gap	Time (sec.)
case9	5297	5289	0.15%	< 1
case14	8082	8065	0.21%	< 1
case30	577	573	0.69%	< 1
case39	41864	41838	0.06%	< 1
case57	41738	41642	0.23%	< 1
case118	129658	129245	0.32%	< 1
case300	722848	717641	0.72%	< 1
case2383wp	1898276	1844550	2.83%	12.6
case2736sp	1308772	1300530	0.63%	14.8

TABLE VIII. OTS POTENTIAL ON CONGESTED BENCHMARKS.

Benchmark	AC-OPF	NF-OTS	Gap	Time (sec.)
case9-c	22081	21359	3.27%	< 1
case14-c	47259	40614	14.06%	< 1
case30-c	1979	1678	15.21%	< 1
case39-c	107536	105490	1.90%	< 1
case57-c	137116	119204	13.06%	< 1
case118-c	2530242	1519930	39.92%	< 1
case300-c	14773480	11886891	19.54%	5.5
case2383wp-c	3612209	3318880	8.12%	15.8
case2736sp-c	3681255	3567490	3.09%	44.7

and the NF-OTS model. The table illustrates that line switching would bring very little benefit in these test cases (the largest potential gain is only 2.83% on case2383wp). This may appear surprising but the example in Section III shows that it is often advantageous to add more lines when there is no congestion in the network, thus reducing the aggregate resistance to transmission. It was also suggested in [4] that AC-OTS brings minimal benefits in networks with no congestion. The MATPOWER benchmarks do not exhibit congestion due to three key factors: (1) the line thermal limits are often assigned to large non-binding values (e.g., 9900 MVA); (2) the generator cost functions are quadratic, which has a tendency to distribute the active power contribution throughout the network; (3) many generators (which are often synchronous condensers in the IEEE specification) have identical cost functions. Overall, it appears that optimal AC-OPF solutions to the MATPOWER cases are also optimal AC-OTS solutions.

Possibly even more surprising is that the NF-OTS formulation, despite its simplicity, is able to provide tight dual bounds on these AC-OTS problems. Recently, convex relaxations [30], [23] had proved the global optimality of AC-OPF solutions for these test cases. Based on the tractable NF-OTS relaxation, Table VII shows that AC-OPF solutions are optimal or near-optimal for the AC-OTS problem as well on these instances.

C. Line Switching on Congested Test Cases

To understand the potential of line switching better, we now create congested variants of the MATPOWER cases. The congested cases are obtained by solving another optimization problem called the AC-MaxFlow. In this formulation, each load is allowed to increase arbitrarily, while maintaining a

TABLE IX. DC-OTS ON THE RTS-96-OTS CASE.

Loading	DC-OPF	DC-OTS	Improvement	Time (sec.)
100%	169844	169016	0.49%	1.93
95%	132635	131999	0.48%	< 1
90%	95521	94993	0.55%	< 1
85%	63689	57988	8.95%	< 1
80%	40407	38239	5.36%	< 1
75%	28471	27394	3.78%	< 1
70%	18256	17307	5.20%	< 1

consistent power factor (i.e., the ratio of active and reactive power remains the same). Additionally, the active and reactive injection capabilities of the generators are set to infinity. The objective is to maximize the loads (i.e., to push as much power through the network as possible). Note that loads that are co-located with generators are removed in order to avoid unbounded solutions and voltage bounds are reduced by 2% before solving the AC-MaxFlow problem to provide power flow flexibility. The resulting AC-MaxFlow solution is then used to build a congested AC-OPF case by fixing the loads values and generation injection limits.

Table VIII shows the results for the congested cases using the AC-OPF and NF-OTS formulations. The gaps between AC-OPF and NF-OTS are now significant, ranging from 1.9% to 39.97%. This is in stark contrast to Table VII but consistent with the case study in Section III. Of course, the NF-OTS relaxation may be very optimistic, which is examined in Section VI.

V. DC-OTS AS AN AC-OTS HEURISTIC

This section investigates whether the DC-OTS formulation can be used as an effective heuristic for the AC-OTS problem. This DC model was used to solve OTS problems in since it is computationally attractive. Indeed, the DC-OTS formulation is a MIP which can be solved by mature optimisation technology [31].

DC-OTS on the 118-OTS-AC Case: When applied to the 118-OTS-AC case, the DC-OPF solution costs \$1484. After an hour, the DC-OTS model yields a solution opening 37 lines and costing \$1373 (a 7.5% improvement). However, an AC-OPF solver using the topology of the DC-OTS solution does not converge to a feasible solution. This is not surprising since the NF-OTS model produces a lower bound of \$1463 to the AC-OTS problem.

DC-OTS on the RTS-96-OTS Case: Table IX summarises the results of the DC-OPF and DC-OTS on the RTS-96-OTS case. On this network, the DC-OTS problem solves very quickly taking only a few seconds. However, the reduction in generation costs varies throughout the different loading values, ranging from 0.49% to 8.95%. AC-OPF feasibility is a major challenging in this network and, similar to [12], we found that only the lowest loading level (i.e. 70%) lead to an AC feasible power flow.

DC-OTS on the Standard Test Cases: It would be premature to draw conclusions about the DC-OTS based on the results of just two power networks. Hence, we evaluate the quality of DC-OTS solutions on the standard test cases. The general procedure for evaluating a DC-OTS solution is as follows: (1) solve the DC-OTS with a time limit of two hours; (2) take the best solution returned by the latter, fixing

TABLE X. FEASIBILITY OF RANDOM MATPOWER CASES TOPOLOGIES.

Case	Samples	DC only	AC only	AC & DC feasible	AC & DC infeasible	Unknown
case9	511	0.00%	0.00%	3.13%	96.67%	0.20%
case14	78392	10.43%	0.00%	27.15%	61.19%	1.23%
case30	45023	18.46%	0.00%	18.77%	56.38%	6.39%
case39	78392	0.87%	0.00%	7.89%	85.79%	5.44%
case57	82488	37.97%	0.00%	8.29%	44.99%	8.75%
case118	92503	2.18%	0.00%	85.81%	11.85%	0.16%
case300	90510	8.84%	0.29%	20.43%	63.51%	7.29%
case2383wp	63261	0.53%	0.00%	31.34%	64.76%	3.37%
case2736sp	65938	1.08%	0.18%	63.45%	34.13%	1.16%

the corresponding topology and solve a QC-OPF problem. If the QC-OPF has no solution, then we have an infeasibility proof for AC-OPF; (3) if QC-OPF returns a solution, we solve the AC-OPF model on the same topology to acquire the true cost of the DC-OTS solution. The DC-OTS projects savings between 1.4% and 6.8%. However, except for case9 and case14, the QC-OPF proves that the DC-OTS topology is infeasible. This further emphasizes that AC feasibility is a major issue for the DC-OTS approximation.

Feasibility of Arbitrary DC-OTS Solutions: The previous section indicated that *high-quality* DC-OTS solutions have problems with feasibility in AC power flow equations. One may wonder if this is a general property for arbitrary network topologies or if it was a special property of high-quality DC-OTS solutions. We investigate this question by producing random sub-graphs of the MATPOWER cases and testing feasibility on each configuration. The testing procedure is as follows: (1) Between 1–10 lines are removed randomly; (2) the resulting sub-network is tested for feasibility using DC-OPF, QC-OPF, and AC-OPF formulations. The solutions are then categorised using the following criteria:

- (DC only), DC-OPF returns a feasible solutions but QC-OPF proves infeasibility
- (AC only), DC-OPF claims infeasibility but AC-OPF returns a feasible solution
- (AC & DC feasible), both DC-OPF and AC-OPF are feasible for the given topology
- (AC & DC infeasible), both DC-OPF and AC-OPF are infeasible for the given topology
- Unknown, both DC-OPF and QC-OPF return a feasible solution but AC-OPF was unable to converge

Table X presents the feasibility results on approximately 60,000–90,000 random topologies per benchmark. The results show that there is a significant percentage of topologies where the DC-OPF is feasible, yet the AC-OPF is not (e.g., 37% on case57). In contrast, almost no topologies are feasible in the AC-OPF and infeasible in the DC-OPF. By comparing Table X to the results of the last subsection, it appears that the high-quality DC-OTS solutions are more likely to be DC only solutions, suggesting high-quality DC-OTS topologies are biased to infeasibility. This is especially the case for case2383wp, where the DC-OTS selects a DC only solution 0.53% of the time.

In summary, it seems evident that OTS studies should avoid using the DC-OTS formulation which exhibits significant feasibility problems on a variety of networks.

TABLE XI. OTS BENEFITS ON THE 118-OTS-AC CASE WHEN NO MORE THAN k LINES ARE OPENED.

k	Cost	Improve.	Time (sec.)	Opened Lines
0	\$1965	0%	0.25	
1	\$1909	2.85%	25	94–96
2	\$1905	3.05%	59	74–75, 94–96
3	\$1889	3.87%	172	76–77, 79–80, 89–91
4	\$1885	4.07%	319	76–77, 79–80, 89–91, 94–95
5	\$1879	4.38%	310	74–75, 76–77, 79–80, 89–91, 94–95
10	\$1866	5.04%	956	—
22+	\$1865	5.14%	1245	—

TABLE XII. AC-OTS ON THE RTS-96-OTS CASE.

Loading	AC-OPF	BONMIN-OTS	Improv.	NF-OTS gap	QC-OTS gap
100%	202222	189989	6.05%	1.76%	0.47%
95%	159069	153306	3.62%	2.07%	0.70%
90%	122205	116651	4.55%	2.31%	0.18%
85%	86698	81241	6.29%	3.60%	0.39%
80%	60976	48456	20.53%	6.86%	2.53%
75%	41103	35239	14.27%	2.64%	0.37%
70%	34582	24368	29.54%	2.41%	0.08%

VI. BONMIN-OTS AS AN AC-OTS HEURISTIC

After noting the unreliable nature of DC-OTS, this section studies the use of BONMIN as an alternate heuristic for the AC-OTS problem. Our primary goal is to assess more accurately the potential benefits of line switching identified by the NF-OTS formulation on the congested test cases.

Bonmin-OTS on the 118-OTS-AC Case: BONMIN-OTS takes 1245 seconds to solve this case study and produces a solution with cost \$1865 by opening 22 lines. This is a 5.14% improvement over the AC-OPF solution, a result similar to the findings of [6] for a comparable 118-bus network. QC-OTS returns a lower bound of \$1585 on this instance, providing an optimality gap of 15.0%. Prior studies on the OTS problem have noticed that a significant part of the cost savings occurs with just one to five lines being removed from the network. Table XI examines this issue with BONMIN-OTS, by evaluating how the objective value evolves as the number of removed lines increases. The results show that a significant portion (2.85%) of the savings occurs after opening one line. However, in contrast to prior studies, our results indicate that turning of many lines (i.e., 10 or more) may yield significant additional gains. Observe also that line 94–96 is selected for $k = \{1, 2\}$ but does not appear in later solutions. This provides an example where an iterative line-opening heuristic (e.g., [6], [18]) can miss the full benefits of AC-OTS.

Bonmin-OTS on the RTS-96-OTS Case: Table XII presents the results of BONMIN-OTS on the RTS-96-OTS case and Table XIII shows the runtime of the algorithms at various loading levels. With cost savings as high as 29.54%, the improvement column indicates that BONMIN-OTS is able to improve generator dispatch significantly through line switching and achieving the benefits suggested in Table VI. Furthermore, a study with the QC-OTS model suggests that the topologies produced by BONMIN-OTS are near optimal (i.e. $< 1\%$ in all but one case).

BONMIN-OTS on Congested Test Cases: Table XIV presents the results of BONMIN-OTS on the congested benchmarks and Table XV shows the runtime of the algorithms. The largest two benchmarks are omitted as they are outside

TABLE XIII. AC-OTS RUNTIME ON THE RTS-96-OTS CASE (SECONDS, WITH A TIME LIMIT OF 36000).

Loading	BONMIN-OTS	QC-OTS
100%	488.49	T.L.
95%	139.13	T.L.
90%	185.20	19295.51
85%	218.63	23554.32
80%	2101.56	T.L.
75%	299.96	3853.57
70%	55.70	383.93

TABLE XIV. AC-OTS ON CONGESTED BENCHMARKS.

Benchmark	AC-OPF	BONMIN-OTS	Improv.	NF-OTS gap	QC-OTS gap
case9-c	22081	22081	0.00%	3.27%	0.00%
case14-c	47259	47259	0.00%	14.06%	3.36%
case30-c	1979	1695	14.3%	1.04%	0.44%
case39-c	107536	107536	0.00%	1.90%	0.45%
case57-c	137116	133473	2.66%	10.7%	3.44%
case118-c	2530242	2478127	2.06%	38.8%	3.58%
case300-c	14773480	14745788	0.19%	19.4%	4.94%

TABLE XV. AC-OTS RUNTIME ON CONGESTED BENCHMARKS (SECONDS, WITH A TIME LIMIT OF 36000).

Benchmark	BONMIN-OTS	QC-OTS
case9-c	< 1	< 1
case14-c	1.42	3.98
case30-c	9.45	16.91
case39-c	8.58	5.32
case57-c	33.7	294.7
case118-c	67.3	15992.1
case300-c	4304.0	T.L.

the scope of the solver. On benchmarks case30-c, case57-c, case118-c, BONMIN-OTS brings improvements between 2% and 14%. On benchmarks case9-c, case14-c and case39-c, BONMIN-OTS produces no improvements and the dual bound from QC-OTS indicates these solutions are near optimal. Note that the QC-OTS dual bounds demonstrate that the BONMIN-OTS solutions are near-optimal on all benchmarks.

Overall, this AC-OTS study indicates that BONMIN-OTS is an effective heuristic for solving the OTS problem, as demonstrated by the QC-OTS’s small duality gaps. The results indicate cost savings ranging from 0%–29% on a variety of congested networks. Although these improvements may appear small in some cases, in an industry that costs over \$500 billion dollars annually in the U.S. [6], even small improvements can lead to significant savings.

VII. CONCLUSION

This paper investigated the benefits of line switching for reducing generator dispatch costs. It studied the properties of various formulations, including the DC power flow model, the QC relaxation, a new NF-OTS relaxation, and a heuristic branch and bound. The key findings are:

- 1) Line switching is mainly beneficial on congested networks and can bring cost reductions up to 29%.
- 2) The DC power flow model is not a reliable method for studying line switching.
- 3) Heuristic branch and bound algorithms can be of great value in producing AC feasible solutions to the OTS problem.
- 4) The QC relaxation is instrumental in providing quality guarantees for primal heuristics.

The potential impact of these results is significant. They indicate that, in practice, black-box MINLP solvers, such as BONMIN, can be an effective solution method for OTS. This is unexpected given the non-convex nature of these problems. It remains to be seen if this solution approach is scalable to models incorporating N-1 reliability constraints [5] and joint unit commitment [11].

Finally, the preliminary cost reduction results presented here are encouraging for OTS. Nevertheless, despite our best efforts to evaluate this problem on a variety of contexts, including light and congested operations, the networks studied by the research community were first published in the sixties and seventies. It remains unclear whether these cost improvements will apply on modern power networks. This advocates for increased collaboration with the power industry, which will hopefully make more network data available for future research in OTS.

ACKNOWLEDGMENT

The authors would like to thank Clayton Barrows for providing the original network data for the 118-OTS and RTS-96-OTS cases. This work was conducted in part at NICTA and is funded by the Australian Government as represented by the Department of Broadband, Communications and the Digital Economy and the Australian Research Council through the ICT Centre of Excellence program.

REFERENCES

- [1] J. Rolim and L. Machado, "A study of the use of corrective switching in transmission systems," *IEEE Transactions on Power Systems*, vol. 14, no. 1, pp. 336–341, 1999.
- [2] W. Shao and V. Vittal, "Corrective switching algorithm for relieving overloads and voltage violations," *IEEE Transactions on Power Systems*, vol. 20, no. 4, pp. 1877–1885, 2005.
- [3] E. Fisher, R. O'Neill, and M. Ferris, "Optimal transmission switching," *IEEE Transactions on Power Systems*, vol. 23, no. 3, pp. 1346–1355, 2008.
- [4] K. Hedman, R. O'Neill, E. Fisher, and S. Oren, "Optimal transmission switching — sensitivity analysis and extensions," *IEEE Transactions on Power Systems*, vol. 23, no. 3, pp. 1469–1479, 2008.
- [5] —, "Optimal transmission switching with contingency analysis," *IEEE Transactions on Power Systems*, vol. 24, no. 3, pp. 1577–1586, aug. 2009.
- [6] T. Potluri and K. Hedman, "Impacts of topology control on the acopf," in *Power and Energy Society General Meeting, 2012 IEEE*, 2012, pp. 1–7.
- [7] J. Fuller, R. Ramasra, and A. Cha, "Fast heuristics for transmission-line switching," *IEEE Transactions on Power Systems*, vol. 27, no. 3, pp. 1377–1386, 2012.
- [8] R. O'Neill, R. Baldick, U. Helman, M. Rothkopf, and J. Stewart, W., "Dispatchable transmission in rto markets," *IEEE Transactions on Power Systems*, vol. 20, no. 1, pp. 171–179, 2005.
- [9] C. Barrows and S. Blumsack, "Transmission switching in the rts-96 test system," *IEEE Transactions on Power Systems*, vol. 27, no. 2, pp. 1134–1135, May 2012.
- [10] C. Barrows, S. Blumsack, and R. Bent, "Computationally efficient optimal transmission switching: Solution space reduction," in *Power and Energy Society General Meeting, 2012 IEEE*, July 2012, pp. 1–8.
- [11] K. Hedman, M. Ferris, R. O'Neill, E. Fisher, and S. Oren, "Co-optimization of generation unit commitment and transmission switching with n-1 reliability," in *Power and Energy Society General Meeting, 2010 IEEE*, July 2010, pp. 1–1.
- [12] C. Barrows, S. Blumsack, and P. Hines, "Correcting optimal transmission switching for ac power flows," in *System Sciences (HICSS), 2014 47th Hawaii International Conference on*, Jan 2014, pp. 2374–2379.
- [13] T. Overbye, X. Cheng, and Y. Sun, "A comparison of the ac and dc power flow models for Imp calculations," in *Proceedings of the 37th Annual Hawaii International Conference on System Sciences*, 2004, p. 9 pp.
- [14] K. Purchala, L. Meeus, D. Van Dommelen, and R. Belmans, "Usefulness of DC power flow for active power flow analysis," *Power Engineering Society General Meeting*, pp. 454–459, 2005.
- [15] B. Stott, J. Jardim, and O. Alsac, "Dc power flow revisited," *IEEE Transactions on Power Systems*, vol. 24, no. 3, pp. 1290–1300, 2009.
- [16] C. Coffrin, P. Van Hentenryck, and R. Bent, "Accurate Load and Generation Scheduling for Linearized DC Models with Contingencies," *Proceedings of the 2012 IEEE Power & Energy Society General Meetings (PES)*, 2012.
- [17] A. Sorokin, S. Rebennack, P. Pardalos, N. Iliadis, and M. Pereira, *Handbook of Networks in Power Systems I: 1 (Energy Systems)*. Springer, 2012.
- [18] M. Soroush and J. Fuller, "Accuracies of optimal transmission switching heuristics based on dcopf and acopf," *IEEE Transactions on Power Systems*, vol. PP, no. 99, pp. 1–9, 2013.
- [19] H. Hijazi, C. Coffrin, and P. V. Hentenryck, "Convex quadratic relaxations of nonlinear programs in power systems," http://www.optimization-online.org/DB_HTML/2013/09/4057.html, 2013.
- [20] P. Bonami, L. T. Biegler, A. R. Conn, G. Cornuejols, I. E. Grossmann, C. D. Laird, J. Lee, A. Lodi, F. Margot, N. Sawaya, and A. Wächter, "An algorithmic framework for convex mixed integer nonlinear programs," *Discrete Optimization*, vol. 5, no. 2, pp. 186 – 204, 2008.
- [21] P. Belotti, "Couenne: a user's manual," Lehigh University, Tech. Rep., 2009. [Online]. Available: <https://projects.coin-or.org/Couenne/browser/trunk/Couenne/doc/couenne-user-manual.pdf?format=raw>
- [22] T. Achterberg, "SCIP: solving constraint integer programs," *Mathematical Programming Computation*, vol. 1, no. 1, pp. 1–41, Jul. 2009. [Online]. Available: <http://dx.doi.org/10.1007/s12532-008-0001-1>
- [23] J. Lavaei and S. Low, "Zero duality gap in optimal power flow problem," *IEEE Transactions on Power Systems*, vol. 27, no. 1, pp. 92 –107, feb. 2012.
- [24] R. Jabr, "Radial distribution load flow using conic programming," *IEEE Transactions on Power Systems*, vol. 21, no. 3, pp. 1458–1459, Aug 2006.
- [25] A. Wächter and L. T. Biegler, "On the implementation of a primal-dual interior point filter line search algorithm for large-scale nonlinear programming," *Mathematical Programming*, vol. 106, no. 1, pp. 25–57, 2006.
- [26] University of Washington, Dept. of Electrical Engineering, "Power systems test case archive," <http://www.ee.washington.edu/research/pstca/>, 1999, accessed: 30/04/2012.
- [27] R. Zimmerman, C. Murillo-S andnchez, and R. Thomas, "Matpower: Steady-state operations, planning, and analysis tools for power systems research and education," *IEEE Transactions on Power Systems*, vol. 26, no. 1, pp. 12 –19, feb. 2011.
- [28] S. Blumsack, "Network topologies and transmission investment under electric-industry restructuring," Ph.D. dissertation, Carnegie Mellon University, 2006.
- [29] C. Grigg, P. Wong, P. Albrecht, R. Allan, M. Bhavaraju, R. Billinton, Q. Chen, C. Fong, S. Haddad, S. Kuruganty, W. Li, R. Mukerji, D. Patton, N. Rau, D. Reppen, A. Schneider, M. Shahidehpour, and C. Singh, "The ieee reliability test system-1996. a report prepared by the reliability test system task force of the application of probability methods subcommittee," *IEEE Transactions on Power Systems*, vol. 14, no. 3, pp. 1010–1020, Aug 1999.
- [30] D. T. Phan, "Lagrangian duality and branch-and-bound algorithms for optimal power flow," *Operations Research*, vol. 60, no. 2, pp. 275–285, 2012.
- [31] R. Bixby, M. Fenelon, Z. Gu, E. Rothberg, and R. Wunderling, *System Modelling and Optimization: Methods, Theory, and Applications*. Kluwer Academic Publishers, 2000, ch. MIP: Theory and practice – closing the gap, pp. 19–49.